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Balance in trees

K.B. Reid, E. DePalma

Department of Mathematics, California State University, San Marcos, CA 92096-0001, USA

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Abstract

We show that the set of vertices in a tree T of smallest weight balance is the (branch weight) centroid. We characterize the set of edges in T with smallest weight-edge difference in terms of the vertices of smallest weight balance. A similar characterization is obtained for the set of edges in T with smallest distance-edge difference in terms of the vertices of smallest distance balance. This yields a new proof that the set of vertices in T of smallest distance balance consists of a single vertex or two adjacent vertices (possibly disjoint from the center and the branch weight centroid).

Keywords: Distance balance; Weight balance; Branch weight centroid

1. Introduction and terminology

Throughout this paper we restrict our attention to finite trees (connected, acyclic, finite graphs). A tree may also be characterized as follows: a graph, G, is a tree if and only if for any two vertices, x, y, in G, there is a unique path in G connecting x and y.

The set of edges of a tree T is denoted as $E(T)$, and the set of vertices as $V(T)$. We will use juxtaposition to indicate an edge connecting vertices x and y, i.e., xy, or equivalently yx, will indicate such an edge. A *branch* of a vertex v is a maximal subtree containing v as an endvertex. The *degree* of a vertex v, denoted $\deg(v)$, is the number of distinct branches of v (i.e., the number of distinct vertices adjacent to v).

The *distance* between two vertices, x, y, of a tree T, denoted $d(x, y)$, is the number of edges of the unique path connecting them. If $x, y \in V(T)$ and $x \neq y$, then $V_{xy} = \{u \in$

E-mail address: breid@csusm.edu (K. Reid).

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$V(T) : d(x,u) < d(y,u)\}$. Note that $V_{xy} \cup V_{yx} = V(T)$ if and only if $d(x,y)$ is odd. In particular, if $xy \in E(T)$, then $V_{xy} \cup V_{yx} = V(T)$ is a partition of $V(T)$.

We utilize special notation throughout the paper as to simplify some expressions of sums of distances.

$$S_x(V_{xy}) = \sum_{z \in V_{xy}} \mathrm{d}(x,z),$$

$$S_y(V_{xy}) = \sum_{z \in V_{xy}} \mathrm{d}(y,z).$$

For example, if T is $K_{1,m}$ with $\deg(x) = m$ and $\deg(y) = 1$, then $S_x(V_{xy}) = m - 1$, while $S_y(V_{xy}) = 2(m-1) + 1 = 2m - 1$.

In this paper we explore two different concepts of "best" balance points of a tree among its vertices (Section 2) and pursue two contrasting concepts among its edges (Section 3). The first concept (Sections 2.1 and 3.1) involves "moments about vertices" and the second concept (Sections 2.2 and 3.2) involves "weights about vertices."

We will make use of the following lemma several times in our discussion. The (easy) proof is left to the reader.

Lemma 1. *Suppose $r_1, r_2, \ldots, r_q$ are q positive real numbers with $r_q > r_1 + \cdots + r_{q-1}$. Let $A \cup B$ be a partition of $\{1, 2, \ldots, q\}$. Then $0 < r_q - (r_1 + \cdots + r_{q-1}) \leqslant |\sum_{i \in A} r_i - \sum_{j \in B} r_j|$.*

2. Vertex balance

2.1. Distance balance of a vertex

Let T be a tree. Consider an arbitrary vertex $x \in V(T)$. We wish to partition the branches of x into two sets that induce two connected components in order to produce a minimum of the absolute value of the difference between the sum of the distances from x to the vertices in each component. This difference is denoted as the *distance balance* of the vertex x,

$$\mathrm{dbal}(x) = \min \left\{ \left| \sum_{z \in V(T_1)} \mathrm{d}(x,z) - \sum_{w \in V(T_2)} \mathrm{d}(x,w) \right| : T_1, T_2 \text{ subtrees of } T \right. \\ \left. \text{such that } V(T_1) \cup V(T_2) = V(T),\ V(T_1) \cap V(T_2) = \{x\} \right\}. \quad (1)$$

Considering the vertices in different branches as weights on different lever arms, we have balanced the tree as best possible about the vertex x when determining $\mathrm{dbal}(x)$. The set of vertices which have the minimum distance balance, $\mathrm{DBV}(T)$, might be considered as the best vertex approximations to the center-of-mass points for our tree. We refer to $\mathrm{DBV}(T)$ as the set of *distance balance vertices* of T. In [3] these were referred to as the set of *balance vertices* of T.

Using double orientations of edges Reid [3] showed that $\mathrm{DBV}(T)$ consists of a single vertex or two adjacent vertices, a result that will also follow later in this paper by different methods. That result was recently reproved by different methods by Shan and Kang [5].

We now prove a useful lemma concerning the distance balance of a vertex.

Lemma 2. *Let x be a vertex of a tree T. Suppose that y, y' are distinct vertices adjacent to x such that $S_x(V_{yx}) \leqslant S_x(V_{y'x})$. If $z \in V_{yx}$, then $\mathrm{dbal}(\cdot)$ is a strictly increasing function on the path P from x to z.*

Proof. Let x, y, y' and z be vertices as in the hypothesis. We use induction on $n \equiv d(x, z)$.

For $n = 1$, $z \in V_{yx}$ implies that $z = y$. Then,

$$\begin{aligned} S_y(V_{xy}) - S_y(V_{yx}) &= [S_x(V_{xy}) + |V_{xy}|] - [S_x(V_{yx}) - |V_{yx}|] \\ &= S_x(V_{xy}) - S_x(V_{yx}) + |V(T)| \\ &\geqslant (S_x(V_{y'x}) - S_x(V_{yx})) + |V(T)|. \end{aligned}$$

The expression in parentheses is nonnegative by the hypothesis of the lemma. So, the value $S_y(V_{xy}) - S_y(V_{yx})$ is positive and, consequently, is $\mathrm{dbal}(y)$, since other partitions of branches about y would only increase this value, as in Lemma 1. Now, since the expression in parentheses is at least the minimum in the definition of $\mathrm{dbal}(x)$, we have $\mathrm{dbal}(y) > \mathrm{dbal}(x)$.

Now suppose $n = d(x, z) > 1$. Consider the path of length n from x to z, say $x = y_0$, $y = y_1, y_2, \ldots, y_n = z$, and assume $\mathrm{dbal}(y_{n-1}) > \mathrm{dbal}(y_{n-2}) > \cdots \mathrm{dbal}(y_1) > \mathrm{dbal}(x)$. Then,

$$S_{y_{n-1}}(V_{y_{n-2}y_{n-1}}) > S_{y_{n-1}}(V_{y'x}) > S_x(V_{y'x}) \geqslant S_x(V_{yx}),$$

where the last inequality follows from the hypothesis of the lemma. Since

$$S_x(V_{yx}) > S_x(V_{y_n y_{n-1}}) > S_{y_{n-1}}(V_{y_n y_{n-1}})$$

we have $S_{y_{n-1}}(V_{y_{n-2}y_{n-1}}) > S_{y_{n-1}}(V_{y_n y_{n-1}})$. Since this inequality implies the inequality in the hypothesis (with the role of x played by y_{n-1}, the role of y played by y_n, and the role of y' played by y_{n-2}), by the argument in the case $n = 1$ above, $\mathrm{dbal}(y_n) > \mathrm{dbal}(y_{n-1})$. Consequently, $\mathrm{dbal}(\cdot)$ is a strictly increasing function from x to $y_n = z$. By induction, the result follows. $\square$

2.2. Weight balance of a vertex

For an arbitrary vertex $x \in V(T)$, we wish to partition the branches of x into two sets that induce two connected components in order to produce a minimum of the absolute value of the difference between the number of vertices in each component. This is similar to the previous definition (1) of the distance balance of a vertex, except that now the distance of the vertices from x is irrelevant and only the number of vertices in each component, thought of as the "weight" of the component, is relevant. Hence, we denote this difference as the *weight balance* of the vertex x and write

$$\begin{aligned} \mathrm{wbal}(x) = \min\{||V(T_1)| - |V(T_2)|| : &\ T_1, T_2 \text{ subtrees of } T \\ &\text{such that } V(T_1) \cup V(T_2) = V(T),\ V(T_1) \cap V(T_2) = \{x\}\}. \end{aligned} \tag{2}$$

We will show that the set of vertices which have the minimum weight balance, denoted $\mathrm{WBV}(T)$ and referred to as the set of *weight balanced vertices* of T, coincides with the centroid (and, hence, consists of one vertex or two adjacent vertices). This definition is reminiscent of the definition of the security center of a connected graph G given by Slater [6]. In that paper, for a vertex x in G, $f(x) = \min\{|V_{xy}| - |V_{yx}| : y \in V - \{x\}\}$, and the security center of G, denoted $C(G)$, is the set of vertices of G for which $f(\cdot)$ is a maximum (note that $f(x)$ can be negative). Slater showed [6] that the security center of a tree is the centroid.

The centroid was initially defined as the branch weight (abbreviated bw) centroid [1]. There are several other known characterizations, namely the median [8], the telephone center [2], the accretion center [7], the latency center [4] and, as mentioned above, the security center [6].

We use the characterization given by the bw-centroid: For arbitrary $x \in V(T)$, the *branch weight* of x, denoted $\mathrm{bw}(x)$, is the number of edges in the largest branch of x. Equivalently, $\mathrm{bw}(x) = \max\{|V_{ux}| : u \text{ adjacent to } x\}$. The bw-centroid of T, or centroid of T, denoted $\mathrm{Centroid}(T)$, is the set of vertices with minimum branch weight. That is, $\mathrm{Centroid}(T) = \{u \in V(T) : \mathrm{bw}(u) \leqslant \mathrm{bw}(y) \forall y \in V(T)\}$. It is well known [1] that this set consists of either a single vertex or two adjacent vertices.

For the following two lemmas, let xy be an edge in a tree T and choose notation so that the $n = \deg(x)$ vertices $u_1, \ldots, u_n$ adjacent to x satisfy $X_1 \leqslant X_2 \leqslant \cdots \leqslant X_n$, where for $1 \leqslant i \leqslant n$, $X_i = |V_{u_i x}|$. Note that $y = u_h$ for some h, $1 \leqslant h \leqslant n$. Additionally, choose notation so that the $m = \deg(y)$ vertices $v_1, \ldots, v_m$ adjacent to y satisfy $Y_1 \leqslant Y_2 \leqslant \cdots \leqslant Y_m$, where for $1 \leqslant i \leqslant m$, $Y_i = |V_{v_i y}|$. Note that $x = v_{h'}$ for some h', $1 \leqslant h' \leqslant m$. Also note that $\mathrm{bw}(x) = X_n$ and $\mathrm{bw}(y) = Y_m$.

Lemma 3. *Let xy be an edge in a tree T. Then* $\mathrm{bw}(x) = \mathrm{bw}(y)$ *implies that* $\mathrm{wbal}(x) = \mathrm{wbal}(y)$.

Proof. Pick an edge xy in T, and suppose that $\mathrm{bw}(x) = \mathrm{bw}(y)$. We actually prove that $\mathrm{wbal}(x) = \mathrm{wbal}(y) = 1$. We use the notation established in the statement prior to the lemma.

If $x \neq v_m$, then $V_{v_m y} \subset V_{yx}$ properly. This implies that $\mathrm{bw}(x) = X_n \geqslant X_h = |V_{u_h x}| = |V_{yx}| > |V_{v_m y}| = Y_m = \mathrm{bw}(y)$, a contradiction. Thus, $x = v_m$. By identical reasoning, $y = u_n$.

Hence, we may write $X_n = |V_{u_n x}| = |V_{yx}| = |\left(\bigcup_{j=1}^{m-1} V_{v_j y}\right) \cup \{y\}| = Y_1 + \cdots + Y_{m-1} + 1$, and similarly $Y_m = X_1 + \cdots + X_{n-1} + 1$.

Now, $\mathrm{bw}(x) = \mathrm{bw}(y)$ implies that $X_n = Y_m$ so that $X_n = X_1 + \cdots + X_{n-1} + 1$ and $Y_m = Y_1 + \cdots + Y_{m-1} + 1$. By the definition of the weight balance of a vertex, (2), and Lemma 1 (where, in Lemma 1, $q = n$, $r_i = X_i$, $1 \leqslant i \leqslant n$, or $q = m$, $r_i = Y_i$, $1 \leqslant i \leqslant m$), these equations, respectively, require

$$\begin{aligned} \mathrm{wbal}(x) &= X_n - (X_1 + \cdots + X_{n-1}) \\ &= 1, \text{ and} \\ \mathrm{wbal}(y) &= Y_m - (Y_1 + \cdots + Y_{m-1}) \\ &= 1. \qquad \square \end{aligned}$$

Lemma 4. *Let xy be an edge in a tree T. Then* $\mathrm{bw}(x) < \mathrm{bw}(y)$ *implies that* $\mathrm{wbal}(x) < \mathrm{wbal}(y)$.

Proof. Pick an edge xy in T and suppose that $\mathrm{bw}(x) < \mathrm{bw}(y)$. Again, adopt the notation established prior to the statement of Lemma 3.

Just as in the proof of Lemma 3, $x = v_m$.

Suppose $y = u_n$. As in the proof of Lemma 3, we may write $X_n = Y_1 + \cdots + Y_{m-1} + 1$ and $Y_m = X_1 + \cdots + X_{n-1} + 1$. Now, $\mathrm{bw}(y) > \mathrm{bw}(x)$ implies that $Y_m > X_n = Y_1 + \cdots + Y_{m-1} + 1$. By the definition of the weight balance of a vertex, (2), and Lemma 1, this equation requires that $\mathrm{wbal}(y) = Y_m - (Y_1 + \cdots + Y_{m-1}) > 1$. Substituting the above equations for Y_m and X_n, this inequality yields $(X_1 + \cdots + X_{n-1} + 1) - (X_n - 1) > 1$, so that

$$0 \leqslant X_1 + \cdots + X_{n-1} - X_n. \tag{3}$$

If $\mathrm{wbal}(x) \geqslant \mathrm{wbal}(y)$, then $\mathrm{wbal}(x) \geqslant Y_m - (Y_1 + \cdots + Y_{m-1}) = 2 + (X_1 + \cdots + X_{n-1}) - X_n$. Combining this with (3), we see that $0 \leqslant X_1 + \cdots + X_{n-1} - X_n < \mathrm{wbal}(x)$. This is a contradiction since $\mathrm{wbal}(x)$ represents the minimum, nonnegative difference of all partitions of the X_i's. Thus, we must have $\mathrm{wbal}(x) < \mathrm{wbal}(y)$.

Now, suppose that $y \neq u_n$, so $y = u_h$ for some fixed $h < n$. Then, as before, $Y_m = X_1 + \cdots + X_{h-1} + X_{h+1} + \cdots + X_n + 1$ and $X_n \geqslant X_h = Y_1 + \cdots + Y_{m-1} + 1$. Since $X_n = \mathrm{bw}(x) < \mathrm{bw}(y) = Y_m$, we see that $Y_1 + \cdots + Y_{m-1} + 1 < Y_m$. By (2) and Lemma 1, this inequality requires that $\mathrm{wbal}(y) = Y_m - (Y_1 + \cdots + Y_{m-1})$. Suppose $\mathrm{wbal}(x) \geqslant \mathrm{wbal}(y)$. This implies $\mathrm{wbal}(x) \geqslant Y_m - (Y_1 + \cdots + Y_{m-1}) = X_1 + \cdots + X_{h-1} + X_{h+1} + \cdots + X_n - X_h + 2$. Since $X_n \geqslant X_h$, we may write $0 \leqslant X_1 + \cdots + X_{h-1} + X_{h+1} + \cdots + X_n - X_h < \mathrm{wbal}(x)$, which is a contradiction to the definition of $\mathrm{wbal}(x)$ as before. Thus, we must have $\mathrm{wbal}(x) < \mathrm{wbal}(y)$. □

Using these two lemmas we now prove the primary result of this section.

Theorem 1. *Let T be a tree. Then*, $\mathrm{WBV}(T) = \mathrm{Centroid}(T)$.

Proof. Let $u \in \mathrm{WBV}(T)$. Suppose u is not in $\mathrm{Centroid}(T)$. Let v be the vertex in the centroid closest to u. It is well known (see Lemma B in the Appendix) that the $\mathrm{bw}(\cdot)$ function is strictly increasing on the path, P, from v to u, and so by Lemma 4, $\mathrm{wbal}(\cdot)$ is also a strictly increasing function on P. This implies that $\mathrm{wbal}(v) < \mathrm{wbal}(u)$, contradicting the assumption that $u \in \mathrm{WBV}(T)$. Thus, $u \in \mathrm{Centroid}(T)$. Therefore, we have shown that $\mathrm{WBV}(T) \subseteq \mathrm{Centroid}(T)$.

If $|\mathrm{Centroid}(T)| = 1$, then clearly $\mathrm{WBV}(T) = \mathrm{Centroid}(T)$.

If $|\mathrm{Centroid}(T)| = 2$, say $\mathrm{Centroid}(T) = \{u, v\}$ with u and v adjacent, then $\mathrm{bw}(u) = bw(v)$. Thus, by Lemma 3, $\mathrm{wbal}(u) = \mathrm{wbal}(v)$. Therefore, $\mathrm{WBV}(T) = \mathrm{Centroid}(T)$. □

3. Edge balance

3.1. Distance balanced edge set

The *distance* of a vertex x, denoted $s(x)$, is defined to be the sum of the distances from x to each vertex in $V(T)$, i.e., $s(x) = \sum_{u \in V(T)} d(x, u)$. For an arbitrary edge $xy \in E(T)$, define the *distance-edge difference* of the edge xy to be

$$s(xy) = |S_x(V_{xy}) - S_y(V_{yx})|. \tag{4}$$

So, $s(xy)$ may be thought of as the positive difference between $s(x)$ computed in the subtree of T induced by V_{xy}, and $s(y)$ computed in the subtree of T induced by V_{yx}.

Denote the set of edges which have the minimum distance-edge difference as $\mathrm{DBE}(T)$. Analogous to the motivation for the distance balance of a vertex, consider attempting to balance a tree at an edge. Namely, given an edge xy, the vertex z with $z \neq x$ or y, is considered as a unit weight on the lever arm whose length is given by the length of the path between z and $\{x, y\}$, i.e., by $d(z, \{x, y\})$. The moment of xy with respect to V_{xy} is $S_x(V_{xy})$ and the moment of xy with respect to V_{yx} is $S_y(V_{yx})$, so $s(xy)$ is a measure of how balanced the edge xy is in T. The edges in $\mathrm{DBE}(T)$ are the "most" balanced edges in T. Therefore, we refer to $\mathrm{DBE}(T)$ as the *distance balanced* edge set, or set of *distance balanced edges*, of T. In [3] these were briefly referred to as the set of *balanced edges* of T. We will show how $\mathrm{DBE}(T)$ can be characterized in terms of $\mathrm{DBV}(T)$, the set of distance balanced vertices. First, we present two lemmas.

Lemma 5. *Let T be a tree with $xy \in E(T)$ and $S_x(V_{xy}) \geqslant S_y(V_{yx})$. If uv is an edge of T with both $u, v \in V_{yx}$, then $s(uv) > s(xy)$.*

Proof. Adjust the notation for u, v so that $d(u, y) < d(v, y)$. Then $V_{xy} \subset V_{uv}$ properly, and $V_{vu} \subset V_{yx}$ properly, which, respectively, imply

$$S_x(V_{xy}) < S_u(V_{uv}) \quad \text{and} \quad S_v(V_{vu}) < S_y(V_{yx}).$$

Consequently, $s(uv) = S_u(V_{uv}) - S_v(V_{vu}) > s(xy)$. □

Lemma 6. *Let T be a tree. If xy, uv are two distinct edges in* $\mathrm{DBE}(T)$, *then xy and uv are adjacent.*

Proof. If there is a vertex $z \neq x$ and z adjacent to y, then we claim that

$$s(yz) = S_y(V_{yz}) - S_z(V_{zy}).$$

To see this, suppose instead that $s(yz) = S_z(V_{zy}) - S_y(V_{yz})$. Then, by Lemma 5, since $x, y \in V_{yz}$ we have $s(xy) > s(yz)$, a contradiction to $xy \in \mathrm{DBE}(T)$.

So, given the form of $s(yz)$, if $ab \in E(T)$, where both $a, b \in V_{zy}$, then by Lemma 5, $s(ab) > s(yz)$. Consequently, ab is not in $\mathrm{DBE}(T)$. In particular, since $uv \in \mathrm{DBE}(T)$, we cannot have both $u, v \in V_{zy}$.

In exactly the same manner, if there is a vertex z' adjacent to x and $z' \neq y$, then we cannot have both $u, v \in V_{z'x}$.

Thus, one of u, v must equal either x or y, which implies that xy and uv are adjacent. □

Lemma 6 restricts the structure of the subtree induced by $\mathrm{DBE}(T)$ to either a single edge or a set of edges all incident with the same vertex. The following theorem relates $\mathrm{DBE}(T)$ to $\mathrm{DBV}(T)$.

Theorem 2. *Let T be a tree. If* $\mathrm{DBE}(T)$ *consists of a single edge, say* $\mathrm{DBE}(T)=\{xy\}$, *then* $\mathrm{DBV}(T)$ *is* $\{x\}$, $\{y\}$ *or* $\{x, y\}$. *If* $\mathrm{DBE}(T)$ *consists of more than one edge, then* $\mathrm{DBV}(T)$ *consists of a single vertex incident with each edge in* $\mathrm{DBE}(T)$.

Proof. Let $xu_1 \in \mathrm{DBE}(T)$ where, without loss of generality throughout the proof,

$$s(xu_1) = S_x(V_{xu_1}) - S_{u_1}(V_{u_1x}). \tag{5}$$

If there is another edge, $uv \in \mathrm{DBE}(T)$, $uv \neq xu_1$, then by Lemma 6, uv is incident with xu_1. However, by Lemma 5, both u and v cannot be in V_{u_1x}. Therefore, both $u, v \in V_{xu_1}$, with one of u, v necessarily equal to x.

Thus, if $u_1, \dots, u_m$ denote the $m \equiv \deg(x)$ distinct vertices adjacent to x, where $xu_1 \in \mathrm{DBE}(T)$ and (5) holds, then we may choose notation so that $\mathrm{DBE}(T) = \{xu_1, \dots, xu_k\}$, where $1 \leqslant k \leqslant m$.

For $2 \leqslant j \leqslant m$, suppose $s(xu_j) = S_{u_j}(V_{u_jx}) - S_x(V_{xu_j})$. Since both $x, u_1 \in V_{xu_j}$, Lemma 5 implies that $s(xu_1) > s(xu_j)$, contrary to $xu_1 \in \mathrm{DBE}(T)$. Now, fix i, $1 \leqslant i \leqslant m$. In conjunction with our initial assumption (5), we have

$$s(xu_i) = S_x(V_{xu_i}) - S_{u_i}(V_{u_ix}). \tag{6}$$

Since $s(xu_i) \geqslant 0$, (6) and $|V_{xu_i}| > 0$ trivially imply that $S_x(V_{xu_i}) + |V_{xu_i}| > S_{u_i}(V_{u_ix})$, or, equivalently,

$$S_{u_i}(V_{xu_i}) > S_{u_i}(V_{u_ix}). \tag{7}$$

Thus, by the definition of the distance balance of a vertex, (1), and Lemma 1, (7) implies

$$\mathrm{dbal}(u_i) = S_{u_i}(V_{xu_i}) - S_{u_i}(V_{u_ix}), \tag{8}$$

since other partitions of branches about u_i would only increase this value.

In particular, Lemma 2 and (7) imply that if $z \in V_{u_ix} - \{u_i\}$, then $\mathrm{dbal}(z) > \mathrm{dbal}(u_i)$.

Using (6), if i and j are distinct indices in $\{1, \dots, m\}$, and $s(xu_i) < s(xu_j)$, then

$$S_x(V_{xu_i}) - S_{u_i}(V_{u_ix}) < S_x(V_{xu_j}) - S_{u_j}(V_{u_jx}).$$

By expanding the summation signs on both sides of this inequality we see that

$$\begin{aligned}&\sum_{p=1, p\neq i}^{m} [(S_{u_p}(V_{u_px})) + |V_{u_px}|] - S_{u_i}(V_{u_ix}) \\ &\quad < \sum_{p=1, p\neq j}^{m} [(S_{u_p}(V_{u_px})) + |V_{u_px}|] - S_{u_j}(V_{u_jx}).\end{aligned}$$

Cancelling like terms we obtain

$$2S_{u_j}(V_{u_jx}) + |V_{u_jx}| < 2S_{u_i}(V_{u_ix}) + |V_{u_ix}|. \tag{9}$$

Similarly, $s(xu_i) = s(xu_j)$ implies equality in (9).

Now fix attention on $j \geqslant 2$. As $xu_1 \in \mathrm{DBE}(T)$, $s(xu_1) \leqslant s(xu_j)$. By (9) and the statement following it,

$$2S_{u_j}(V_{u_jx}) + |V_{u_jx}| \leqslant 2S_{u_1}(V_{u_1x}) + |V_{u_1x}|.$$

Now, $V_{u_1x} \subset V_{xu_j}$ properly, which directly implies that

$$2S_{u_1}(V_{u_1x}) + |V_{u_1x}| < 2S_x(V_{xu_j}) + |V_{xu_j}|.$$

From these last two inequalities we obtain

$$2S_{u_j}(V_{u_jx}) + |V_{u_jx}| < 2S_x(V_{xu_j}) + |V_{xu_j}|,$$

or,

$$S_{u_j}(V_{u_jx}) + |V_{u_jx}| - S_x(V_{xu_j}) < S_x(V_{xu_j}) + |V_{xu_j}| - S_{u_j}(V_{u_jx}).$$

Consequently,

$$S_x(V_{u_jx}) - S_x(V_{xu_j}) < S_{u_j}(V_{xu_j}) - S_{u_j}(V_{u_jx}).$$

However, by (8), the right-hand side of this equation is just $\mathrm{dbal}(u_j)$. Coupled with the fact that

$$\begin{aligned}\mathrm{dbal}(u_j) &= S_{u_j}(V_{xu_j}) - S_{u_j}(V_{u_jx})\\ &= S_x(V_{xu_j}) + |V_{xu_j}| - S_{u_j}(V_{u_jx})\\ &> S_x(V_{xu_j}) - |V_{u_jx}| - S_{u_j}(V_{u_jx})\\ &= S_x(V_{xu_j}) - S_x(V_{u_jx})\end{aligned}$$

we deduce that

$$|S_x(V_{xu_j}) - S_x(V_{u_jx})| < \mathrm{dbal}(u_j). \tag{10}$$

From the definition of the distance balance of a vertex, (1), we must have

$$\mathrm{dbal}(x) \leqslant |S_x(V_{xu_j}) - S_x(V_{u_jx})|.$$

This inequality for $\mathrm{dbal}(x)$ and inequality (10) imply that $\mathrm{dbal}(x) < \mathrm{dbal}(u_j)$. Combined with the statement following (8) we may conclude that for $j \geqslant 2$, if $w \in V_{u_jx}$, then $\mathrm{dbal}(x) < \mathrm{dbal}(w)$. Also, recall that for $i = 1$ in the statement following (8), if $w \in V_{u_1x} - \{u_1\}$, then $\mathrm{dbal}(u_1) < \mathrm{dbal}(w)$. We now consider two separate cases, where $k = 1$ or $k > 1$.

Case 1: $k \equiv |\mathrm{DBE}(T)| = 1$.

(1) If $\mathrm{dbal}(x) < \mathrm{dbal}(u_1)$, then $\mathrm{DBV}(T) = \{x\}$.
(2) If $\mathrm{dbal}(x) = \mathrm{dbal}(u_1)$, then $\mathrm{DBV}(T) = \{x, u_1\}$.
(3) If $\mathrm{dbal}(x) > \mathrm{dbal}(u_1)$, then $\mathrm{DBV}(T) = \{u_1\}$.

Case 2: $k \equiv |\mathrm{DBE}(T)| > 1$.

In the above proof we showed that for $j \neq 1$, $s(xu_1) \leqslant s(xu_j)$ implies that $\mathrm{dbal}(x) < \mathrm{dbal}(u_j)$. Since, in this specific case we now have $k \geqslant 2$, we could have used identical reasoning to show instead that for $j \neq 2$, $s(xu_2) \leqslant s(xu_j)$ implies that $\mathrm{dbal}(x) < \mathrm{dbal}(u_j)$. Therefore, when $k > 1$ we may conclude that $\mathrm{dbal}(x) < \mathrm{dbal}(u_1)$, and, thus, $\mathrm{DBV}(T) = \{x\}$. $\square$

Examples. To show that the different cases at the end of the previous proof can actually occur, we provide specific examples of trees satisfying the initial assumption (5). In addition, in all of the following examples the classical center, the centroid and the set of distance balance vertices are all mutually disjoint. These examples are derivative from [3, Theorem 1]. Let T be a "broom" tree with a "handle" of ten vertices and m vertices as "bristles," i.e., $V(T) = \{v_1, \ldots, v_m, w_0, w_1, \ldots, w_9\}$ and $E(T) = \{v_1w_0, \ldots, v_mw_0, w_0w_1, w_1w_2, \ldots, w_8w_9\}$. For Case 1, w_1 and w_2 will play the role of u_1 and x, respectively. Then $m = 10$ gives (1), $m = 12$ gives (2) and $m = 13$ gives (3). For Case 2, set $m = 21$. In this case w_0, w_1, w_2 play the roles of u_1, x, u_2, respectively.

As a consequence of this theorem, we deduce independently a result in [3] that $\mathrm{DBV}(T)$ must consist of either a single vertex or two adjacent vertices.

3.2. Weight balanced edge set

For an arbitrary edge $xy \in E(T)$, define the *weight-edge difference* of the edge xy to be

$$r(xy) = ||V_{xy}| - |V_{yx}||. \tag{11}$$

In [3, Remark 5], this was simply called the balance (in T) at the edge xy. Denote the set of edges which have the minimum weight-edge difference as $\mathrm{WBE}(T)$. This is similar to the previous definition of the distance balanced edge set, except that now we disregard the distance of the vertices from the edges. Instead, we consider only the number of vertices in the two subtrees obtained by deleting the edge xy, and we think of these numbers as total weights of the two subtrees. This type of balance of the tree about an edge will be best if we choose an edge in $\mathrm{WBE}(T)$. In accordance with the discussion in Section 2.2, we refer to $\mathrm{WBE}(T)$ as the *weight balanced* edge set, or set of *weight balanced edges*, of T. In [3, Remark 5], these edges were simply called balanced edges.

We present the details for the claim made in [3, Remark 5], that $\mathrm{WBE}(T)$ can be related to $\mathrm{WBV}(T)$. This will be done in a similar manner that $\mathrm{DBE}(T)$ is related to $\mathrm{DBV}(T)$ in Theorem 2.

Theorem 3. *Let T be a tree. If* $\mathrm{WBV}(T)$ *consists of two adjacent vertices, say* $\mathrm{WBV}(T) = \{x, y\}$, *then* $\mathrm{WBE}(T) = \{xy\}$. *If* $\mathrm{WBV}(T)$ *consists of a single vertex, say* $\mathrm{WBV}(T) = \{x\}$, *then every edge in* $\mathrm{WBE}(T)$ *is incident with x.*

Proof. We use $\mathrm{WBV}(T) = \mathrm{Centroid}(T)$ (Theorem 1).

Suppose that T contains two (adjacent) centroids x and y. Since $\mathrm{bw}(x) = \mathrm{bw}(y)$, $r(xy) = ||V_{yx}| - |V_{xy}|| = 0$. So, every edge in $\mathrm{WBE}(T)$ must have $r(\cdot)$ value 0. Suppose wz is an edge with both w and z in V_{xy}. We may suppose that $d(x,w) < d(x,z)$. By Lemma B in the Appendix bw$(\cdot)$ is strictly increasing on the path from x to z, so bw$(w) <$ bw(z). Thus, by Lemma A in the Appendix, $|V_{wz}| > |V_{zw}|$. This implies that $r(wz) > 0$, so wz is not in WBE(T). Similarly, no edge with both ends in V_{yx} is in WBE(T). Consequently, WBE$(T) = \{xy\}$.

Suppose that T has a single centroid. Now, if $\{x, xy, y, yz, z\}$ is a path in T and bw$(x) >$ bw$(y) >$ bw(z), then since V_{xy} is properly contained in V_{yz} and V_{zy} is properly contained in V_{yx}, $r(xy) = ||V_{xy}| - |V_{yx}|| = |V_{yx}| - |V_{xy}| > |V_{zy}| - |V_{xy}| > |V_{zy}| - |V_{yz}| = ||V_{zy}| - |V_{yz}|| = r(yz)$. Thus, by Lemma B in the Appendix, as the function bw$(\cdot)$ strictly increases on the vertices along any path in T originating at the centroid vertex, the function $r(\cdot)$ strictly increases on the edges of any such path. Consequently, any edge of least weight-edge difference must be adjacent to the centroid. $\square$

Example. For each k, $1 \leqslant k \leqslant (n-1)/2$ or $k = n-1$, there is a tree T with n vertices, exactly one vertex x in WBV(T) and exactly k edges in *WBE*(T). For example, when $k = n-1$, use $K_{1,n-1}$. When $1 \leqslant k \leqslant (n-1)/2$, write $n-1 = kq + r$, $0 \leqslant r < k$. Then $q \geqslant 2$. Form k copies of $K_{1,q-1}$, with vertex x_i of degree $q-1$ in the ith copy. Let T be the tree with n vertices formed from these k copies and a new vertex x of degree $k + r$ adjacent to $x_1, \ldots, x_k$ and r new vertices $y_1, \ldots, y_r$.

Note that when $(n-1)/2 < k < n-1$, then there is no tree T of order n with $|\text{WBE}(T)| = k$. To see this, suppose there is such a tree T. By Theorem 3, all k edges in WBE(T) are incident to a single vertex, say WBE$(T) = \{xx_1, \ldots, xx_k\}$. Since T contains only $n-1-k$ other edges, and $n-1-k < k$, at least one of $x_1, \ldots, x_k$, say x_j, has degree 1. As $r(xx_j) = n-2$ and $xx_j \in$ WBE(T), $r(xx_i) = n-2$ for all i, $1 \leqslant i \leqslant k$. Consequently, deg$(x_i) = 1$ for all i, $1 \leqslant i \leqslant k$. Now, T is not $K_{1,n-1}$ (since $|\text{WBE}(K_{1,n-1})| = n-1 > k$), so T contains some vertex $y \neq x$ with deg$(y) > 1$. If zw denotes any edge on the path between y and x, then $r(zw) < n-2 = r(xx_1)$, contrary to $xx_1 \in$ WBE(T).

We can also state Theorem 3 in an equivalent way that is parallel to the form of the statement of Theorem 2. Namely,

Let T be a tree. If WBE(T) consists of a single edge, say WBE$(T) = \{xy\}$, then WBV(T) is $\{x\}$, $\{y\}$ or $\{x, y\}$. If WBE(T) consists of more than one edge, then WBV(T) consists of a single vertex incident with each edge in WBE(T).

The straightforward proof is left to the reader.

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Appendix

For reference in the text and for completeness we include two basic facts.

Lemma A. *Let x and y be adjacent vertices in a tree such that* $\mathrm{bw}(x)\geqslant\mathrm{bw}(y)$. *Then* $\mathrm{bw}(x)=|V_{yx}|\geqslant|V_{xy}|$, *with* $|V_{yx}|>|V_{xy}|$ *if* $\mathrm{bw}(x)>\mathrm{bw}(y)$.

Proof. Suppose instead that $\mathrm{bw}(x)=|V_{zx}|$ for some z adjacent to x and $z\neq y$. Since $V_{zx}\subset V_{xy}$ properly, we have $\mathrm{bw}(x)=|V_{zx}|<|V_{xy}|\leqslant\mathrm{bw}(y)$, a contradiction. Also, since by definition $\mathrm{bw}(y)\geqslant|V_{xy}|$, we see that $\mathrm{bw}(x)\geqslant\mathrm{bw}(y)$ also implies $|V_{yx}|\geqslant|V_{xy}|$, where strict inequality in the first equation implies strict inequality in the second. $\square$

Lemma B. *If x and y are adjacent vertices in a tree T, and* $\mathrm{bw}(x)\geqslant\mathrm{bw}(y)$, *then* $\mathrm{bw}(\cdot)$ *is a strictly increasing function on the path, P, from x to any vertex* $v\in V_{xy}-\{x\}$.

Proof. Let x and y be adjacent vertices in a tree. Then by Lemma A, $\mathrm{bw}(y)=|V_{yx}|$. Let $v\in V_{xy}-\{x\}$. We use induction on $n\equiv d(x,v)$.

If $n=1$, then $\mathrm{bw}(v)\geqslant|V_{xv}|>|V_{yx}|=\mathrm{bw}(x)$.

For $n>1$, let v' be on P, v' adjacent to v. Also let v'' be on P, v'' adjacent to v', $v''\neq v$. Since $d(x,v')=n-1$, by the inductive hypothesis $\mathrm{bw}(\cdot)$ is strictly increasing on P from x to v'. Thus, $\mathrm{bw}(v')>\mathrm{bw}(v'')$, and by Lemma A, $\mathrm{bw}(v')=|V_{v''v'}|$. Since $V_{v''v'}\subset V_{v'v}$ properly, we see that $|V_{v'v}|>|V_{v''v'}|$. Thus, $\mathrm{bw}(v)\geqslant|V_{v'v}|>|V_{v''v'}|=\mathrm{bw}(v')$. So, $\mathrm{bw}(\cdot)$ is strictly increasing on P. $\square$

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